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Abstract. The sum of a supermodular function, assumed nondecreasing in the choice variable, and of a "concavely supermodularizable" function, assumed nonincreasing in the parameter variable, satisfies the Milgrom-Shannon (1994, Monotone comparative statics, *Econometrica* 62, 157-180) single crossing condition. As an application, I prove existence of a pure strategy Nash equilibrium in a Cournot duopoly with logconcave demand, affiliated types, and nondecreasing costs.

Keywords. Supermodularity; separable objective function; single crossing condition; quantity competition.

JEL-Codes. C02, C72, C65.

1. Introduction

Monotone comparative statics has proven to be a useful tool in numerous economic applications (Topkis, 1978; Milgrom and Roberts, 1990). Over time, cardinal notions of supermodularity have been refined into more general, ordinal variants (Milgrom and Shannon, 1994). However, ordinal techniques are less straightforward to apply when the problem is characterized by a separable objective function. For instance, while the sum of two supermodular functions is again supermodular, the sum of two logsupermodular functions need not be logsupermodular.[1]

In this paper, I show that the sum of a supermodular function, assumed to be monotone increasing in the choice variable, and a "concavely supermodularizable" function, assumed to be monotone decreasing in the parameter variable, satisfies the Milgrom-Shannon single crossing condition. As an application, I prove existence of an isotone equilibrium in a Cournot duopoly with logconcave demand, affiliated types, and nondecreasing costs. This is the first result of this sort that both makes assumptions solely on primitives of the model and works without negative prices.

The rest of the paper is structured as follows. Section 2 defines and characterizes the notion of concavely increasing differences. The key result of the paper is stated and proved in Section 3. Section 4 contains the application to the Cournot model.

2. Concavely increasing differences

This section introduces the notion of concavely increasing differences and

[1] Cf. Athey (2002).

offers a characterization that will be useful to prove the key result of this paper. I will make use of the standard terminology and notation introduced in Milgrom and Shannon (1994). Definitions and clarifications of the basic concepts should be sought there.

Definition 1. *Let X be a lattice, and T be a partially ordered set. A function $g : X \times T \to \mathbb{R}$ has [strict]* ***concavely increasing differences in*** *$(x;t)$ if for any $x' > x''$ and $t' > t''$,*

$$\phi(g(x',t')) - \phi(g(x'',t')) \underset{[>]}{\geq} \phi(g(x',t'')) - \phi(g(x'',t'')) \tag{1}$$

for some strictly increasing, concave transformation ϕ.[2]

Obviously, any function with increasing differences has concavely increasing differences. Moreover, any function that is logsupermodular on the product space $X \times T$ has concavely increasing differences.[3] Note, however, that there are functions that fulfill neither property and still satisfy Definition 1. Indeed, in Example 1, the inequality $\phi(19) - \phi(29) \geq \phi(18) - \phi(27)$ fails for transformations $\phi(y) = y$ and $\phi(y) = \ln y$, yet holds for $\phi(y) = -1/y$. Example 1 here

Definition 1 allows formulating sufficient conditions on a separable objective function to satisfy the Milgrom-Shannon single crossing property. To prepare the key result, I state the following characterization of concavely increasing differences.

Lemma 1. *A function g has [strict] concavely increasing differences in $(x;t)$ if and only if for any $x' > x''$ and $t' > t''$ such that* $\min\{g(x'',t'), g(x',t'')\} \geq$

[2] The transformation ϕ may, but need not depend on the vector (x',x'',t',t''). Moreover, the domain of the transformation must be some interval.

[3] Another example are rootsupermodular functions, as defined by Eeckhout and Kircher (forthcoming).

$\min\{g(x'',t''), g(x',t')\}$, *the inequality*

$$g(x',t') - g(x'',t') \underset{[>]}{\geq} g(x',t'') - g(x'',t'') \tag{2}$$

holds.

Proof. The proof is given for the case of nonstrict differences only. The other case is completely analogous. "Only if". Assume g has concavely increasing differences in $(x;t)$, and fix $x' > x''$ and $t' > t''$. Write $a = g(x'',t'')$, $b = g(x'',t')$, $c = g(x',t'')$, and $d = g(x',t')$. Then there is a strictly increasing, concave transformation ϕ such that $\phi(d) - \phi(b) \geq \phi(c) - \phi(a)$. Consider $\min\{b,c\} \geq \min\{a,d\}$, say $a \leq b \leq c$. Then necessarily $c \leq d$, because ϕ is strictly increasing and $\phi(a) + \phi(d) \geq \phi(b) + \phi(c)$. To prove (2), assume to the contrary that $d - b < c - a$, so that with a concave and strictly increasing ϕ, $\phi(d) - \phi(b) < \phi(c) - \phi(a)$, a contradiction. "If". Using the same notation, assume g satisfies $d - b \geq c - a$ provided that $\min\{b,c\} \geq \min\{a,d\}$. I need to find a strictly increasing, concave ϕ such that $\phi(b) + \phi(c) \leq \phi(a) + \phi(d)$. Consider first $\min\{b,c\} < \min\{a,d\}$. If $b = c$, the claim follows for any strictly increasing ϕ. Therefore, without loss of generality, $b < a, c, d$. In this case, define $\phi(a) = a$, $\phi(c) = c$, $\phi(d) = d$, and $\phi(b)$ sufficiently negative so that $\phi(b) + \phi(c) \leq \phi(a) + \phi(d)$. Clearly, ϕ can be extended to a strictly increasing, concave function on $\mathbb{R}$. Consider now $\min\{b,c\} \geq \min\{a,d\}$. Then, by assumption, $b + c \leq a + d$. Clearly, in this case, ϕ can be chosen linear. $\square$

Thus, concavely increasing differences requires increasing differences only when "off-diagonal entries do not undercut the minimum diagonal entry." For illustration, note that Lemma 1 implies that any monotone function, either increasing or decreasing, that exhibits concavely increasing differences must have increasing differences.

3. Separable objective functions

The key result of the paper is the following.

Theorem 1. *Let X be a lattice, and T be a partially ordered set. Consider functions $g, h : X \times T \to \mathbb{R}$. Assume that g has concavely increasing differences in $(x;t)$ and is nondecreasing [nonincreasing] in t. Assume also that h has increasing differences in $(x;t)$ and is nonincreasing [nondecreasing] in x. Then $g+h$ satisfies the single crossing condition in $(x;t)$. If, in addition, g has strict concavely increasing differences and is strictly increasing [decreasing] in t, then $g+h$ satisfies the strict single crossing property in $(x;t)$.*

Proof. According to the definition, $f = g + h$ satisfies the single crossing property in $(x;t)$ if for $x' > x''$ and $t' > t''$, $f(x',t'') \geq f(x'',t'')$ implies that $f(x',t') \geq f(x'',t')$ and $f(x',t'') > f(x'',t'')$ implies that $f(x',t') > f(x'',t')$. So take arbitrary $x' > x''$ and $t' > t''$. Impose

$$f(x',t'') \geq f(x'',t''). \tag{3}$$

Since h is nonincreasing in x, inequality (3) implies $g(x',t'') \geq g(x'',t'')$. Moreover, g is nondecreasing in t, so $g(x'',t') \geq g(x'',t'')$. By assumption, g has concavely increasing differences. Thus, by Lemma 1,

$$g(x',t') - g(x'',t') \geq g(x',t'') - g(x'',t''). \tag{4}$$

But h has increasing differences in $(x;t)$, so that

$$h(x',t') - h(x'',t') \geq h(x',t'') - h(x'',t''). \tag{5}$$

Adding (4) and (5) term by term yields

$$f(x',t') - f(x'',t') \geq f(x',t'') - f(x'',t''). \tag{6}$$

Combining this with (3), one obtains

$$f(x',t') - f(x'',t') \geq 0. \tag{7}$$

as desired. Moreover, if inequality (3) holds strictly, so does (7). This proves the claim for nonstrict differences. To prove the claim also for strict differences, note that inequality (4) is then strict, so that inequality (3) implies the strict version of (7), as required by the strict single crossing property. □

For intuition, focus on T and X being two-element subsets of $\mathbb{R}$, and ϕ being the logarithm. Clearly, the conclusion is obvious when g has actually increasing differences. So assume that the slope $\frac{g(x',t)-g(x'',t)}{x'-x''}$, regarded as a function of t, strictly decreases, while the ratio $\frac{g(x',t)}{g(x'',t)}$ weakly increases in t, as illustrated in Figure 1. Since g is nondecreasing in t, a moment's reflection shows that this is possible only when g is strictly downward-sloping at t''.[4] But then, adding a function h that is nonincreasing in x implies the single crossing condition for the sum.

Figure 1 here

To extend Theorem 1, re-order T (or, equivalently, X). E.g., assume that g has concavely decreasing differences in $(x;t)$, which is defined in analogy to Definition 1, and that g is nonincreasing [nondecreasing] in t. Then, with h having decreasing differences in $(x;t)$ and being nonincreasing [nondecreasing] in x, it follows that $g+h$ satisfies the dual single crossing condition in $(x;t)$.

[4]Indeed, if g were upwards sloping or flat at t'', then the strictly lower slope at t' would make the ratio $g(x',t)/g(x'',t)$ decline strictly in t.

Another extension assumes that g has convexly increasing or decreasing differences in $(x;t)$, where again, the notions are defined in analogy to Definition 1. For instance, when g has convexly increasing differences in $(x;t)$ and is nondecreasing [nonincreasing] in t, and h has increasing differences in $(x;t)$ and is nondecreasing [nonincreasing] in x, then $g+h$ satisfies the single crossing property.[5]

Theorem 1 is readily applied to the comparative statics of optimization problems and noncooperative games using the results in Milgrom and Shannon (1994).[6] For instance, Amir's (1996) main result follows directly from Theorem 1. But Theorem 1 can also be fruitfully applied to Bayesian games, as illustrated in the next section.

4. Application

This section deals with equilibrium existence in the Cournot duopoly with affiliated types. A pure strategy Nash equilibrium is known to exist regardless of distributional assumptions provided certainty payoffs are submodular in firm's actions (Vives, 1990). Under additional complementarities between actions and types, even an isotone equilibrium exists for affiliated types (Athey, 2001, McAdams, 2003, Van Zandt and Vives, 2007). However, cardinal submodularity in actions is not a completely innocuous assumption in the Bayesian Cournot model because uncertainty then tends to generate negative prices (Einy et al., 2010).[7]

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[5]Indeed, it is not difficult to check that g has convexly increasing differences in $(x;t)$ if and only if $-g$ has concavely decreasing differences in $(x;t)$, so that the claim follows from the first extension.

[6]Only when the choice set is not a chain, each term needs to be supermodular in the choice variable to ensure the sum is quasisupermodular in x.

[7]Indeed, Cournot profits that are submodular in actions imply Novshek's (1985) mar-

Addressing this issue, I will show now that a logconcave inverse demand and nondecreasing costs are sufficient for the existence of an isotone equilibrium in a duopoly with affiliated costs. In contrast to prior results, negative prices do not obtain even though all assumptions are on the primitives of the model.

Inverse demand is given by a nonincreasing function p, assumed to be nonnegative and logconcave. There are two firms $i = 1, 2$, each receiving a private signal t_i, referred to as the firm's type, and drawn from a compact interval $T_i \subset \mathbb{R}$. Types are *inversely affiliated*, i.e., jointly distributed according to some a.e. logsubmodular density.[8] Each firm i produces output $x_i \geq 0$ at costs $C_i(x_i, t)$, where $t = (t_i, t_{-i})$. Costs are assumed nondecreasing and continuous in output. Moreover, marginal costs are nonincreasing in own type, while (potentially) nondecreasing in the other firm's type. It is assumed that there is an output level $\overline{x}$ above which no firm has an incentive to operate.[9] Denote firm i's strategy by $\xi_i = \xi_i(t_i)$. Expected profits of a firm i of type t_i producing output x_i read

$$f_i(x_i, t_i) = \int x_i p(x_i + \xi_{-i}(t_{-i})) - C_i(x_i, t) d\Phi_i(t_{-i}|t_i), \tag{8}$$

where Φ_i denotes the conditional distribution function of t_{-i} given t_i. It is claimed that f_i has the single crossing property in $(x_i; t_i)$ provided ξ_{-i} is

ginal revenue condition on inverse demand. The marginal revenue condition, in turn, can be seen to be equivalent to inverse demand being a concave function of log-output. Hence, if inverse demand is declining somewhere, it must eventually cause negative prices.

[8] The density is assumed bounded and atomless, to ensure that expected revenues and costs exist and are finite for all type subintervals and for all nondecreasing strategies of the rival.

[9] For instance, assume that $x_i p(x_i) - C_i(x_i, t) < -C_i(0, t)$ for all $i = 1, 2$, $t \in T_i \times T_{-i}$, and $x_i > \overline{x}$.

monotone increasing. For this, write $f_i = g_i + h_i$, where

$$g_i(x_i, t_i) = \int x_i p(x_i + \xi_{-i}(t_{-i})) d\Phi_i(t_{-i}|t_i) \text{ and} \tag{9}$$

$$h_i(x_i, t_i) = -\int C_i(x_i, t) d\Phi_i(t_{-i}|t_i), \tag{10}$$

respectively, denote expected revenues and (negatively signed) expected costs. To apply Theorem 1, note that ex post revenues $x_i p(x_i + x_{-i})$ are logsubmodular in (x_i, x_{-i}). Hence, because ξ_{-i} is monotone increasing, $x_i p(x_i + \xi_{-i}(t_{-i}))$ is logsubmodular in (x_i, t_{-i}). Therefore, with inversely affiliated types, g_i is log-supermodular in (x_i, t_i).[10] Moreover, as p is nonincreasing, ξ_{-i} is monotone increasing, and types are inversely affiliated, it follows that g_i is nondecreasing in t_i. Consider now the cost term. By assumption, $C_i(x_i, t)$ is submodular in (x_i, t_i) and supermodular in (x_i, t_{-i}). As types are inversely affiliated, it follows that h_i is supermodular in (x_i, t_i).[11] Furthermore, since costs are nondecreasing in output, h_i is nonincreasing in x_i. Thus, f_i satisfies the single crossing condition in $(x_i; t_i)$ for any nondecreasing ξ_{-i}. It follows now from Corollary 2.1 in Athey (2001) that an isotone pure strategy Nash equilibrium exists.

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[10] See the discussion following Lemma 2 in Athey (2002).

[11] Cf. Fact (v) in Athey (2001, p. 872).

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x	*t*	
	0	1
0	27	29
1	18	19

Example 1 − The tabulated function *g* fails to satisfy increasing differences in *(x;t)* and logsupermodularity, but has concavely increasing differences in *(x;t)*.

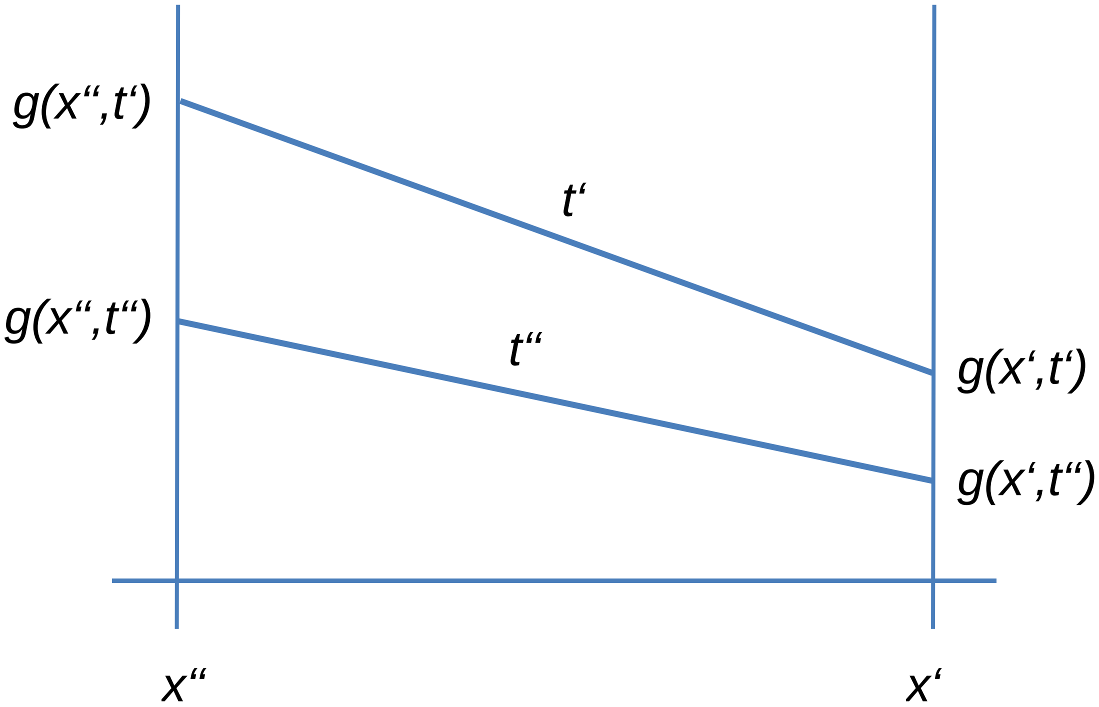

Figure 1 − If the slope of *g* with respect to *x* strictly decreases in *t*, the ratio *g(x‘,t)/g(x“,t)* weakly increases in *t*, and *g* is monotone increasing in *t*, then *g* must be downwards sloping at *t“*.